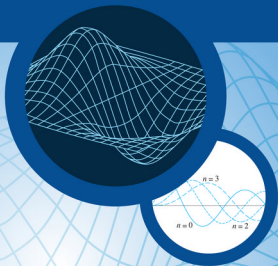


# Mathematics for Physicists

B. R. Martin

G. Shaw



WILEY



# Mathematics for Physicists

---

# The Manchester Physics Series

*General Editors*

**J.R. FORSHAW, H.F. GLEESON, F.K. LOEBINGER**

*School of Physics and Astronomy,  
University of Manchester*

---

<b>Properties of Matter</b>	B.H. Flowers and E. Mendoza
<b>Statistical Physics</b> <i>Second Edition</i>	F. Mandl
<b>Electromagnetism</b> <i>Second Edition</i>	I.S. Grant and W.R. Phillips
<b>Statistics</b>	R.J. Barlow
<b>Solid State Physics</b> <i>Second Edition</i>	J.R. Hook and H.E. Hall
<b>Quantum Mechanics</b>	F. Mandl
<b>Computing for Scientists</b>	R.J. Barlow and A.R. Barnett
<b>The Physics of Stars</b> <i>Second Edition</i>	A.C. Phillips
<b>Nuclear Physics</b>	J.S. Lilley
<b>Introduction to Quantum Mechanics</b>	A.C. Phillips
<b>Particle Physics</b> <i>Third Edition</i>	B.R. Martin and G. Shaw
<b>Dynamics and Relativity</b>	J.R. Forshaw and A.G. Smith
<b>Vibrations and Waves</b>	G.C. King
<b>Mathematics for Physicists</b>	B.R. Martin and G. Shaw

# Mathematics for Physicists

**B.R. MARTIN**

Department of Physics and Astronomy  
University College London

**G. SHAW**

Department of Physics and Astronomy  
Manchester University

**WILEY**

This edition first published 2015  
© 2015 John Wiley & Sons, Ltd

*Registered office*

John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at [www.wiley.com](http://www.wiley.com).

The right of the author to be identified as the author of this work has been asserted in accordance with the Copyright, Designs and Patents Act 1988.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by the UK Copyright, Designs and Patents Act 1988, without the prior permission of the publisher.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

Designations used by companies to distinguish their products are often claimed as trademarks. All brand names and product names used in this book are trade names, service marks, trademarks or registered trademarks of their respective owners. The publisher is not associated with any product or vendor mentioned in this book.

**Limit of Liability/Disclaimer of Warranty:** While the publisher and author have used their best efforts in preparing this book, they make no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. It is sold on the understanding that the publisher is not engaged in rendering professional services and neither the publisher nor the author shall be liable for damages arising herefrom. If professional advice or other expert assistance is required, the services of a competent professional should be sought.

The advice and strategies contained herein may not be suitable for every situation. In view of ongoing research, equipment modifications, changes in governmental regulations, and the constant flow of information relating to the use of experimental reagents, equipment, and devices, the reader is urged to review and evaluate the information provided in the package insert or instructions for each chemical, piece of equipment, reagent, or device for, among other things, any changes in the instructions or indication of usage and for added warnings and precautions. The fact that an organization or Website is referred to in this work as a citation and/or a potential source of further information does not mean that the author or the publisher endorses the information the organization or Website may provide or recommendations it may make. Further, readers should be aware that Internet Websites listed in this work may have changed or disappeared between when this work was written and when it is read. No warranty may be created or extended by any promotional statements for this work. Neither the publisher nor the author shall be liable for any damages arising herefrom.

*Library of Congress Cataloging-in-Publication Data.*

Martin, B. R. (Brian Robert), author.

Mathematics for physicists / B.R. Martin, G. Shaw.

pages cm

Includes bibliographical references and index.

ISBN 978-0-470-66023-2 (cloth) – ISBN 978-0-470-66022-5 (pbk.) 1. Mathematics. 2. Mathematical physics. I. Shaw, G. (Graham), 1942– author. II. Title.

QC20.M35 2015

510–dc23

2015008518

Set in 11/13pt Computer Modern by Aptara Inc., New Delhi, India.

---

# Contents

---

<i>Editors' preface to the Manchester Physics Series</i>	xi
<i>Authors' preface</i>	xiii
<i>Notes and website information</i>	xv
<b>1 Real numbers, variables and functions</b>	<b>1</b>
1.1 Real numbers	1
1.1.1 Rules of arithmetic: rational and irrational numbers	1
1.1.2 Factors, powers and rationalisation	4
*1.1.3 Number systems	6
1.2 Real variables	9
1.2.1 Rules of elementary algebra	9
*1.2.2 Proof of the irrationality of $\sqrt{2}$	11
1.2.3 Formulas, identities and equations	11
1.2.4 The binomial theorem	13
1.2.5 Absolute values and inequalities	17
1.3 Functions, graphs and co-ordinates	20
1.3.1 Functions	20
1.3.2 Cartesian co-ordinates	23
Problems 1	28
<b>2 Some basic functions and equations</b>	<b>31</b>
2.1 Algebraic functions	31
2.1.1 Polynomials	31
2.1.2 Rational functions and partial fractions	37
2.1.3 Algebraic and transcendental functions	41
2.2 Trigonometric functions	41
2.2.1 Angles and polar co-ordinates	41
2.2.2 Sine and cosine	44
2.2.3 More trigonometric functions	46
2.2.4 Trigonometric identities and equations	48
2.2.5 Sine and cosine rules	51
2.3 Logarithms and exponentials	53
2.3.1 The laws of logarithms	54
2.3.2 Exponential function	56
2.3.3 Hyperbolic functions	60
2.4 Conic sections	63
Problems 2	68

<b>3</b>	<b>Differential calculus</b>	<b>71</b>
3.1	Limits and continuity	71
3.1.1	Limits	71
3.1.2	Continuity	75
3.2	Differentiation	77
3.2.1	Differentiability	78
3.2.2	Some standard derivatives	80
3.3	General methods	82
3.3.1	Product rule	83
3.3.2	Quotient rule	83
3.3.3	Reciprocal relation	84
3.3.4	Chain rule	86
3.3.5	More standard derivatives	87
3.3.6	Implicit functions	89
3.4	Higher derivatives and stationary points	90
3.4.1	Stationary points	92
3.5	Curve sketching	95
	Problems 3	98
<b>4</b>	<b>Integral calculus</b>	<b>101</b>
4.1	Indefinite integrals	101
4.2	Definite integrals	104
4.2.1	Integrals and areas	105
4.2.2	Riemann integration	108
4.3	Change of variables and substitutions	111
4.3.1	Change of variables	111
4.3.2	Products of sines and cosines	113
4.3.3	Logarithmic integration	115
4.3.4	Partial fractions	116
4.3.5	More standard integrals	117
4.3.6	Tangent substitutions	118
4.3.7	Symmetric and antisymmetric integrals	119
4.4	Integration by parts	120
4.5	Numerical integration	123
4.6	Improper integrals	126
4.6.1	Infinite integrals	126
4.6.2	Singular integrals	129
4.7	Applications of integration	132
4.7.1	Work done by a varying force	132
4.7.2	The length of a curve	133
*4.7.3	Surfaces and volumes of revolution	134
*4.7.4	Moments of inertia	136
	Problems 4	137
<b>5</b>	<b>Series and expansions</b>	<b>143</b>
5.1	Series	143
5.2	Convergence of infinite series	146



5.3	Taylor's theorem and its applications	149
5.3.1	Taylor's theorem	149
5.3.2	Small changes and l'Hôpital's rule	150
5.3.3	Newton's method	152
*5.3.4	Approximation errors: Euler's number	153
5.4	Series expansions	153
5.4.1	Taylor and Maclaurin series	154
5.4.2	Operations with series	157
*5.5	Proof of d'Alembert's ratio test	161
*5.5.1	Positive series	161
*5.5.2	General series	162
*5.6	Alternating and other series	163
	Problems 5	165
<b>6</b>	<b>Complex numbers and variables</b>	<b>169</b>
6.1	Complex numbers	169
6.2	Complex plane: Argand diagrams	172
6.3	Complex variables and series	176
*6.3.1	Proof of the ratio test for complex series	179
6.4	Euler's formula	180
6.4.1	Powers and roots	182
6.4.2	Exponentials and logarithms	184
6.4.3	De Moivre's theorem	185
*6.4.4	Summation of series and evaluation of integrals	187
	Problems 6	189
<b>7</b>	<b>Partial differentiation</b>	<b>191</b>
7.1	Partial derivatives	191
7.2	Differentials	193
7.2.1	Two standard results	195
7.2.2	Exact differentials	197
7.2.3	The chain rule	198
7.2.4	Homogeneous functions and Euler's theorem	199
7.3	Change of variables	200
7.4	Taylor series	203
7.5	Stationary points	206
*7.6	Lagrange multipliers	209
*7.7	Differentiation of integrals	211
	Problems 7	214
<b>8</b>	<b>Vectors</b>	<b>219</b>
8.1	Scalars and vectors	219
8.1.1	Vector algebra	220
8.1.2	Components of vectors: Cartesian co-ordinates	221
8.2	Products of vectors	225
8.2.1	Scalar product	225
8.2.2	Vector product	228

8.2.3	Triple products	231
*8.2.4	Reciprocal vectors	236
8.3	Applications to geometry	238
8.3.1	Straight lines	238
8.3.2	Planes	241
8.4	Differentiation and integration	243
	Problems 8	246
<b>9</b>	<b>Determinants, Vectors and Matrices</b>	<b>249</b>
9.1	Determinants	249
9.1.1	General properties of determinants	253
9.1.2	Homogeneous linear equations	257
9.2	Vectors in $n$ Dimensions	260
9.2.1	Basis vectors	261
9.2.2	Scalar products	263
9.3	Matrices and linear transformations	265
9.3.1	Matrices	265
9.3.2	Linear transformations	270
9.3.3	Transpose, complex, and Hermitian conjugates	273
9.4	Square Matrices	274
9.4.1	Some special square matrices	274
9.4.2	The determinant of a matrix	276
9.4.3	Matrix inversion	278
9.4.4	Inhomogeneous simultaneous linear equations	282
	Problems 9	284
<b>10</b>	<b>Eigenvalues and eigenvectors</b>	<b>291</b>
10.1	The eigenvalue equation	291
10.1.1	Properties of eigenvalues	293
10.1.2	Properties of eigenvectors	296
10.1.3	Hermitian matrices	299
*10.2	Diagonalisation of matrices	302
*10.2.1	Normal modes of oscillation	305
*10.2.2	Quadratic forms	308
	Problems 10	312
<b>11</b>	<b>Line and multiple integrals</b>	<b>315</b>
11.1	Line integrals	315
11.1.1	Line integrals in a plane	315
11.1.2	Integrals around closed contours and along arcs	319
11.1.3	Line integrals in three dimensions	321
11.2	Double integrals	323
11.2.1	Green's theorem in the plane and perfect differentials	326
11.2.2	Other co-ordinate systems and change of variables	330
11.3	Curvilinear co-ordinates in three dimensions	333
11.3.1	Cylindrical and spherical polar co-ordinates	334

---

11.4	Triple or volume integrals	337
11.4.1	Change of variables	338
	Problems 11	340
<b>12</b>	<b>Vector calculus</b>	<b>345</b>
12.1	Scalar and vector fields	345
12.1.1	Gradient of a scalar field	346
12.1.2	Div, grad and curl	349
12.1.3	Orthogonal curvilinear co-ordinates	352
12.2	Line, surface, and volume integrals	355
12.2.1	Line integrals	355
12.2.2	Conservative fields and potentials	359
12.2.3	Surface integrals	362
12.2.4	Volume integrals: moments of inertia	367
12.3	The divergence theorem	368
12.3.1	Proof of the divergence theorem and Green's identities	369
*12.3.2	Divergence in orthogonal curvilinear co-ordinates	372
*12.3.3	Poisson's equation and Gauss' theorem	373
*12.3.4	The continuity equation	376
12.4	Stokes' theorem	377
12.4.1	Proof of Stokes' theorem	378
*12.4.2	Curl in curvilinear co-ordinates	380
*12.4.3	Applications to electromagnetic fields	381
	Problems 12	384
<b>13</b>	<b>Fourier analysis</b>	<b>389</b>
13.1	Fourier series	389
13.1.1	Fourier coefficients	390
13.1.2	Convergence	394
13.1.3	Change of period	398
13.1.4	Non-periodic functions	399
13.1.5	Integration and differentiation of Fourier series	401
13.1.6	Mean values and Parseval's theorem	405
13.2	Complex Fourier series	407
*13.2.1	Fourier expansions and vector spaces	409
13.3	Fourier transforms	410
13.3.1	Properties of Fourier transforms	414
*13.3.2	The Dirac delta function	419
*13.3.3	The convolution theorem	423
	Problems 13	426
<b>14</b>	<b>Ordinary differential equations</b>	<b>431</b>
14.1	First-order equations	433
14.1.1	Direct integration	433
14.1.2	Separation of variables	434
14.1.3	Homogeneous equations	435

14.1.4	Exact equations	438
14.1.5	First-order linear equations	440
14.2	Linear ODEs with constant coefficients	441
14.2.1	Complementary functions	442
14.2.2	Particular integrals: method of undetermined coefficients	446
*14.2.3	Particular integrals: the $D$ -operator method	448
*14.2.4	Laplace transforms	453
*14.3	Euler's equation	459
	Problems 14	461
<b>15</b>	<b>Series solutions of ordinary differential equations</b>	<b>465</b>
15.1	Series solutions	465
15.1.1	Series solutions about a regular point	467
15.1.2	Series solutions about a regular singularity: Frobenius method	469
15.1.3	Polynomial solutions	475
15.2	Eigenvalue equations	478
15.3	Legendre's equation	481
15.3.1	Legendre functions and Legendre polynomials	482
*15.3.2	The generating function	487
*15.3.3	Associated Legendre equation	490
*15.3.4	Rodrigues' formula	492
15.4	Bessel's equation	494
15.4.1	Bessel functions	495
*15.4.2	Properties of non-singular Bessel functions $J_\nu(x)$	499
	Problems 15	502
<b>16</b>	<b>Partial differential equations</b>	<b>507</b>
16.1	Some important PDEs in physics	510
16.2	Separation of variables: Cartesian co-ordinates	511
16.2.1	The wave equation in one spatial dimension	512
16.2.2	The wave equation in three spatial dimensions	515
16.2.3	The diffusion equation in one spatial dimension	518
16.3	Separation of variables: polar co-ordinates	520
16.3.1	Plane-polar co-ordinates	520
16.3.2	Spherical polar co-ordinates	524
16.3.3	Cylindrical polar co-ordinates	529
*16.4	The wave equation: d'Alembert's solution	532
*16.5	Euler equations	535
*16.6	Boundary conditions and uniqueness	538
*16.6.1	Laplace transforms	540
	Problems 16	544
	<i>Answers to selected problems</i>	549
	<i>Index</i>	559

---

# Editors' preface to the Manchester Physics Series

---

The Manchester Physics Series is a set of textbooks at first degree level. It grew out of the experience at the University of Manchester, widely shared elsewhere, that many textbooks contain much more material than can be accommodated in a typical undergraduate course; and that this material is only rarely so arranged as to allow the definition of a short self-contained course. The plan for this series was to produce short books so that lecturers would find them attractive for undergraduate courses, and so that students would not be frightened off by their encyclopaedic size or price. To achieve this, we have been very selective in the choice of topics, with the emphasis on the basic physics together with some instructive, stimulating and useful applications.

Although these books were conceived as a series, each of them is self-contained and can be used independently of the others. Several of them are suitable for wider use in other sciences. Each Author's Preface gives details about the level, prerequisites, etc., of that volume.

The Manchester Physics Series has been very successful since its inception over 40 years ago, with total sales of more than a quarter of a million copies. We are extremely grateful to the many students and colleagues, at Manchester and elsewhere, for helpful criticisms and stimulating comments. Our particular thanks go to the authors for all the work they have done, for the many new ideas they have contributed, and for discussing patiently, and often accepting, the suggestions of the editors.

Finally, we would like to thank our publisher, John Wiley & Sons, Ltd., for their enthusiastic and continued commitment to the Manchester Physics Series.

J. R. Forshaw  
H. F. Gleeson  
F. K. Loebinger  
*August 2014*



---

## Authors' preface

---

Our aim in writing this book is to produce a relatively short volume that covers all the essential mathematics needed for a typical first degree in physics, from a starting point that is compatible with modern school mathematics syllabuses. Thus, it differs from most books, which include many advanced topics, such as tensor analysis, group theory, etc., that are not required in a typical physics degree course, except as specialised options. These books are frequently well over a thousand pages long and contain much more material than most undergraduate students need. In addition, they are often not well interfaced with school mathematics and start at a level that is no longer appropriate. Mathematics teaching at schools has changed over the years and students now enter university with a wide variety of mathematical backgrounds.

The early chapters of the book deliberately overlap with senior school mathematics, to a degree that will depend on the background of the individual reader, who may quickly skip over those topics with which he or she is already familiar. The rest of the book covers the mathematics that is usually compulsory for all students in their first two years of a typical university physics degree, plus a little more. Although written primarily for the needs of physics students, it would also be appropriate for students in other physical sciences, such as astronomy, chemistry, earth science, etc.

We do not try to cover all the more advanced, optional courses taken by some physics students, since these are already well treated in more advanced texts, which, with some degree of overlap, take up where our book leaves off. The exception is statistics. Although this is required by undergraduate physics students, we have not included it because it is usually taught as a separate topic, using one of the excellent specialised texts already available.

The book has been read in its entirety by one of the editors of the Manchester Physics Series, Jeff Forshaw of Manchester University, and we are grateful to him for many helpful suggestions that have improved the presentation.

B.R. Martin  
G. Shaw  
*April 2015*





---

## Notes and website information

---

### **‘Starred’ material**

Some sections of the book are marked with a star. These contain more specialised or advanced material that is not required elsewhere in the book and may be omitted at a first reading.

### **Website**

Any misprints or other necessary corrections brought to our attention will be listed on [www.wiley.com/go/martin/mathsforphysicists](http://www.wiley.com/go/martin/mathsforphysicists). We would also be grateful for any other comments about the book.

### **Examples, problems and solutions**

Worked examples are given in all chapters. They are an integral part of the text and are designed to illustrate applications of material discussed in the preceding section. There is also a set of problems at the end of each chapter. Some equations which are particularly useful in problem solving are highlighted in the text for ease of access and brief ‘one-line’ answers to most problems are given at the end of the book, so that readers may quickly check whether their own answer is correct. Readers may access the full solutions to all the odd-numbered problems at [www.wiley.com/go/martin/mathsforphysicists](http://www.wiley.com/go/martin/mathsforphysicists). Full solutions to all problems are available to instructors at the same website, which also contains electronic versions of the figures.



# 1

---

## Real numbers, variables and functions

---

In this chapter we introduce some simple ideas about real numbers, i.e. the ordinary numbers used in arithmetic and measurements, real variables and algebraic functions of a single variable. This discussion will be extended in Chapter 2 by considering some important examples in more detail: polynomials, trigonometric functions, exponentials, logarithms and hyperbolic functions. Much of the material in these first two chapters will probably already be familiar to many readers and so is covered briefly, but even if this is the case, it is useful revision and sets the scene for later chapters.

### 1.1 Real numbers

This section starts from the basic rules of arithmetic and introduces a number of essential techniques for manipulating real numerical quantities. We also briefly consider number systems other than the decimal system.

#### 1.1.1 Rules of arithmetic: rational and irrational numbers

The first contact with mathematics is usually via counting, using the positive integers  $1, 2, 3, 4, \dots$  (also called *natural numbers*). Later, fractional numbers such as  $\frac{1}{2}, \frac{3}{5}$ , etc. and negative numbers  $-1, -3, -\frac{1}{3}, -\frac{7}{9}$ , etc. are introduced, together with the rules for combining positive and negative numbers and the basic laws of arithmetic. As we will build on these laws later in this chapter, it is worth reminding oneself of what they are by stating them in a somewhat formal way as follows.

(i) *Commutativity*: The result of subtracting or dividing two integers is dependent on the order in which the operations are performed, but addition and multiplication are independent of the order. For example,

$$3 + 6 = 6 + 3 \quad \text{and} \quad 3 \times 6 = 6 \times 3, \quad (1.1a)$$

but

$$5 - 3 \neq 3 - 5 \quad \text{and} \quad 5 \div 3 \neq 3 \div 5, \quad (1.1b)$$

where  $\neq$  means *not equal to*.

(ii) *Associativity*: The result of subtracting or dividing three or more integers is dependent on the way the integers are associated, but is independent of the association for addition and multiplication. Examples are

$$(2 + 3) + 4 = 2 + (3 + 4) \quad \text{and} \quad 2 \times (3 \times 4) = (2 \times 3) \times 4, \quad (1.2a)$$

but

$$6 - (3 - 2) \neq (6 - 3) - 2 \quad \text{and} \quad 12 \div (6 \div 2) \neq (12 \div 6) \div 2. \quad (1.2b)$$

(iii) *Distributivity*: Multiplication is distributed over addition and subtraction from both left and right, whereas division is only distributed over addition and subtraction from the right. For example, for multiplication:

$$2 \times (4 + 3) = (2 \times 4) + (2 \times 3) \quad (1.3a)$$

and

$$(3 - 2) \times 4 = (3 \times 4) - (2 \times 4), \quad (1.3b)$$

but for division, from the right we have

$$(60 + 15) \div 3 = (60 \div 3) + (15 \div 3), \quad (1.3c)$$

whereas division from the left gives

$$60 \div (12 + 3) = 60 \div 15 \neq (60 \div 12) + (60 \div 3). \quad (1.3d)$$

Positive and negative integers and fractions can all be expressed in the general form  $n/m$ , where  $n$  and  $m$  are integers (with  $m \neq 0$  because division by zero is undefined). A number of this form is called a *rational number*. The latter is said to be *proper* if its numerator is less than its denominator, otherwise it is said to be *improper*. The operations of addition, subtraction, multiplication and division, when applied to rational numbers, always result in another rational number. In the case of fractions, multiplication is applied to the

numerators and denominators separately; for division, the fraction is inverted and then multiplied. Examples are

$$\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28} \quad \text{and} \quad \frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}. \quad (1.4a)$$

For addition (and subtraction) all the terms must be taken over a common denominator. An example is:

$$\frac{3}{4} - \frac{5}{7} + \frac{1}{3} = \frac{(3 \times 3 \times 7) - (5 \times 3 \times 4) + (1 \times 4 \times 7)}{3 \times 4 \times 7} = \frac{31}{84}. \quad (1.4b)$$

Not all numbers can be written in the form  $n/m$ . The exceptions are called *irrational numbers*. Examples are the square root of 2, that is,  $\sqrt{2} = 1.414\dots$ , and the ratio of the circumference of a circle to its diameter, that is,  $\pi = 3.1415926\dots$ , where the dots indicate a non-recurring sequence of numbers. Irrational numbers, when expressed in decimal form, always lead to such non-recurrence sequences, but even rational numbers when expressed in this form may not always terminate, for example  $\frac{2}{11} = 0.1818\dots$ . The proof that a given number is irrational can be very difficult, but is given for one particularly simple case in Section 1.2.2.

In practice, an irrational number may be represented by a rational number to any accuracy one wishes. Thus  $\pi$  is often represented as  $\frac{22}{7} = 3.143$  in rough calculations, or as  $\frac{355}{113} = 3.141593$  in more accurate work. Rational and irrational numbers together make up the class of so-called *real numbers* that are themselves part of a larger class of numbers called *complex numbers* that we will meet in Chapter 6. It is worth remarking that infinity, denoted by the symbol  $\infty$ , is not itself a real number. It is used to indicate that a quantity may become arbitrarily large.

In the examples above of irrational numbers, the sequence of numbers after the decimal point is endless and so, in practice, one has to decide where to terminate the string. This is called *rounding*. There are two methods of doing this: quote either the *number of significant figures* or the *number of decimal places*. Consider the number 1234.567\dots To two decimal places this is 1234.57; the last figure has been rounded up to 7 because the next number in the string after 6 is 7, which is greater than 5. Likewise, we would round down if the next number in the string were less than 5. If the next number in the string were 5, then the 5 and the next number following it are rounded up or down to the nearest even multiple of 10, and the zero dropped. For example, 1234.565 to two decimal places would be 1234.56, whereas 1234.575 would be rounded to 1234.58. If we were to quote 1234.567 to five significant figures, it would be 1234.6 and to three significant figures it would be 1230.

### 1.1.2 Factors, powers and rationalisation

Integer numbers may often be represented as the product of a number of smaller integers. This is an example of a process called *factorisation*, that is, decomposition into a product of smaller terms, or *factors*. For example, 24 is equal to  $2 \times 2 \times 2 \times 3$ . In this example, the integers in the product cannot themselves be factorised further. Such integers are called *prime numbers*. (By convention, unity is not considered a prime number.) By considering all products of the prime numbers in the factorisation, we arrive at the result that the *factors* of 24 are 1, 2, 3, 4, 6, 8, 12 and 24, that is, these are all the numbers that divide exactly into 24. If we have several numbers, the *highest common factor* (HCF) is the largest factor that can divide exactly into all the numbers. The *lowest common multiple* (LCM) is the smallest number into which all the given numbers will divide exactly. Thus the HCF of 24 and 36 is 12 and the LCM of all three numbers is 72.

In the example of the factorisation of the number 24 above, the factor 2 occurs three times. It is common to encounter situations where a number is multiplied by itself several times. A convenient notation for this is to introduce the idea of a *power* (or *index*)  $n$ , such that, for example,  $5^n \equiv 5 \times 5 \times 5 \dots n \text{ times}$ . To emphasise that this relation *defines* the index  $n$ , the usual two-line equality sign has been replaced by a three-line equality sign ( $\equiv$ ). So, using powers, we could also write 24 in the compact prime-number factorised form  $24 = 3 \times 2^3$ . Any real number  $p$  to power zero is *by definition* equal to unity, that is,  $p^0 \equiv 1$  for any  $p$ .

By writing out in full, it is easy to see that multiplying the *same* integers each raised to a power is equivalent to adding the powers. Thus

$$\begin{aligned} 5^n \times 5^m &= (5 \times 5 \times 5 \dots n \text{ times}) \times (5 \times 5 \times 5 \dots m \text{ times}) \\ &= 5 \times 5 \times 5 \dots (n + m) \text{ times} \\ &= 5^{(n+m)} \end{aligned} \quad (1.5a)$$

and analogously for division,

$$5^n / 5^m = 5^{(n-m)}. \quad (1.5b)$$

A power can also be a fraction or rational number, since, for example, the combination rule (1.5a) implies  $5^{1/2} \times 5^{1/2} = 5^1$ , so that  $5^{1/2} = \sqrt{5}$ . Similarly the expression  $3^{1/3} 3^0 3^{4/3} / 27^{1/3}$ , for example, can be simplified to give

$$3^{1/3} 3^0 3^{4/3} / 27^{1/3} = 3^{(1/3+0+4/3-1)} = 3^{2/3}. \quad (1.5c)$$

An example of the use of factors is to express numbers in so-called *scientific notation* (also called *normal form*). In this representation,

any real number is written as the product of a number between  $-10$  and  $+10$  (excluding the numbers  $\pm 10$  themselves), with as many decimal places as required, and a power of 10. The number 1245.678 to four significant figures in scientific notation is therefore  $1.246 \times 10^3$ .

It is conventional to write arithmetical forms in a compact form and to remove as far as possible fractional powers from the denominator of a fraction, a process called *rationalisation*. For example, consider the form

$$\frac{1}{2\sqrt{5}+1} - \frac{1}{2\sqrt{5}+2}. \quad (1.6a)$$

By taking the terms over a common denominator and then multiplying numerator and denominator by  $(11 - 3\sqrt{5})$ , we have

$$\begin{aligned} \frac{1}{2\sqrt{5}+1} - \frac{1}{2\sqrt{5}+2} &= \frac{(2\sqrt{5}+2) - (2\sqrt{5}+1)}{(2\sqrt{5}+1)(2\sqrt{5}+2)} = \frac{1}{2(11+3\sqrt{5})} \\ &= \frac{(11-3\sqrt{5})}{2(11+3\sqrt{5})(11-3\sqrt{5})} = \frac{11-3\sqrt{5}}{152}, \end{aligned} \quad (1.6b)$$

which is the rationalised form of (1.6a).

### Example 1.1

Simplify the following forms:

$$(a) \left(7\frac{1}{9}\right)^{3/2}, \quad (b) 5^{1/3}25^{-1/2}/5^{-1/6}5^{-2/3}, \quad (c) 3^{1/2}27^{-2/3}9^{-1/2}.$$

### Solution

$$(a) \left(7\frac{1}{9}\right)^{3/2} = (64/9)^{3/2} = (8/3)^3 = 512/27,$$

$$(b) 5^{1/3}25^{-1/2}/5^{-1/6}5^{-2/3} = 5^{(1/3-1+1/6+2/3)} = 5^{1/6},$$

$$(c) 3^{1/2}27^{-2/3}9^{-1/2} = 3^{1/2}3^{-2}3^{-1} = 3^{-5/2} = 3^{2/5}.$$

### Example 1.2

Rationalise the numerical expressions:

$$(a) \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{3}+1}, \quad (b) \frac{1}{\sqrt{2}+1}, \quad (c) \frac{(2-\sqrt{5})}{(3+2\sqrt{5})}.$$

### Solution

(a) Taking both terms over a common denominator gives

$$\frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}+1 + \sqrt{3}-1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \sqrt{3}.$$

(b) Multiplying numerator and denominator by  $(\sqrt{2} - 1)$  gives

$$\frac{1}{\sqrt{2} + 1} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \sqrt{2} - 1.$$

(c) Multiplying numerator and denominator by  $(3 - 2\sqrt{5})$  gives

$$\frac{(2 - \sqrt{5})}{(3 + 2\sqrt{5})} = \frac{(2 - \sqrt{5})(3 - 2\sqrt{5})}{(3 + 2\sqrt{5})(3 - 2\sqrt{5})} = \frac{7\sqrt{5} - 16}{11}.$$

### \*1.1.3 Number systems<sup>1</sup>

All the numbers in the previous sections are expressed in the decimal system, where the ‘basis’ set of integers is  $0, 1, 2, \dots, 9$ . Real numbers in this system are said to be to ‘*base 10*’. In the number 234, for example, the integers 2, 3 and 4 indicate how many powers of 10 are present, reading from the right, i.e.

$$234 = (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0). \quad (1.7)$$

Any other base could equally well be used and in some circumstances other number systems are more appropriate. The most widely used number system other than base 10 is the *binary system*, based on the two integers 0 and 1, that is, *base 2*, so we will only discuss this case. Its importance stems from its use in computers, because the simplest electrical switch has just two states, ‘open’ and ‘closed’. To distinguish numbers in this system we will write them with a subscript 2.

As an example, consider the number 123. In the binary system this is  $1111011_2$ . To check: in the decimal system,

$$1111011_2 = 2^6 + 2^5 + 2^4 + 2^3 + 2^1 + 2^0 = 123. \quad (1.8)$$

Fractions are accommodated by using negative values for the indices. Thus the number 6.25 in the binary system is  $110.01_2$ . To check: in the decimal system,

$$110.01_2 = 2^2 + 2^1 + 2^{-2} = 4 + 2 + 0.25 = 6.25. \quad (1.9)$$

To convert a number in one basis to another is straightforward, if rather tedious. Consider, for example, the conversion of the number 51.78 to the binary system. We start with the integer 51 and find the largest value of an integer  $n$  such that  $2^n$  is less than or equal to

<sup>1</sup>The reader is reminded that the results of starred sections are not needed later, except in other starred sections, and therefore they may prefer to omit them on a first reading.



51 and then note the remainder  $R = 51 - 2^n$ . This is then repeated by again finding the largest number  $n$  such that  $2^n$  is less than or equal to  $R$ , and continued in this way until the remainder is zero. We thus obtain:

$$51 = 2^5 + 19 = 2^5 + 2^4 + 3 = 2^5 + 2^4 + 2^1 + 1 = 2^5 + 2^4 + 2^1 + 2^0,$$

so that in the binary system

$$51 = 110011_2. \quad (1.10a)$$

Similarly, we can convert the numbers after the decimal point using negative powers. This gives

$$0.78 = 2^{-1} + 0.28 = 2^{-1} + 2^{-2} + 0.03 \approx 2^{-1} + 2^{-2} + 2^{-5},$$

so again in the binary system,

$$0.78 \approx 0.11001_2 \quad (1.11a)$$

and finally,

$$51.78 = 110011.11001_2, \quad (1.11b)$$

in the binary system, which represents the decimal number to an accuracy of two decimal places.

All the normal arithmetic operations of addition, subtraction, multiplication and division can be carried out in any number system. For example, in the binary system, we have the basic result  $1_2 + 1_2 = 10_2$ . So adding the numbers  $101_2$  and  $1101_2$  gives  $101_2 + 1101_2 = 10010_2$ . To check, we can again use the decimal system. Thus,

$$101_2 = 2^2 + 2^0 = 5, \text{ and } 1101_2 = 2^3 + 2^2 + 2^0 = 13, \quad (1.12a)$$

with

$$10010_2 = 2^4 + 2^1 = 18. \quad (1.12b)$$

As an example of multiplication, consider the numbers 5 and 7. In the binary system these are  $101_2$  and  $111_2$ , respectively, and multiplying them together gives, using  $1_2 + 1_2 = 10_2$ ,

$$\begin{array}{r} 101 \\ 111 \\ \hline 10100 \\ 1010 \\ 101 \\ \hline 100011 \end{array}$$

Once again, we can check the result using the decimal system:

$$100011_2 = 2^5 + 2^1 + 2^0 = 35. \quad (1.13)$$

As an example of division, consider the numbers 51 and 3. In the binary system these are  $110011_2$  and  $11_2$ , respectively, and dividing them we have

$$\begin{array}{r} 10001 \\ 11 \overline{)110011} \\ \underline{110000} \\ 11 \\ \underline{11} \\ 00 \end{array}$$

So the quotient is  $10001_2$ , which in the decimal system is  $2^4 + 2^0 = 17$ , as required.

### **Example 1.3**

Write the decimal number 100 in base 3 and base 4.

#### **Solution**

(a) The decimal number 100 written as powers of 3 is  $100 = 3^4 + (2 \times 3^2) + 3^0$ , so to base 3 it is  $10201_3$ .

(b) The decimal number 100 written as powers of 4 is  $100 = 4^3 + (2 \times 4^2) + 4^1$ , so to base 4 it is  $1210_4$ .

### **Example 1.4**

Consider the base 3 numbers  $p = 201_3$  and  $q = 112_3$ . Find (a)  $p + q$ , (b)  $p - q$ , (c)  $p \times q$  and (d)  $p/q$  to two decimal places and check your results in the decimal system.

#### **Solution**

In base 3,

(a)  $p + q = 201_3 + 112_3 = 1020_3$ , which as a decimal number is  $3^3 + (2 \times 3) = 33$ ,

(b)  $p - q = 201_3 - 112_3 = 12_3$ , which as a decimal number is  $3^1 + (2 \times 3^0) = 5$ ,

(c)  $p \times q = 201_3 \times 112_3 = 100212_3$ , which as a decimal number is  $3^5 + (2 \times 3^2) + 3^1 + (2 \times 3^0) = 266$ ,

(d)  $p/q = 201_3/112_3 = 1.1002_3 \dots$ , which as a decimal number is  $3^0 + 3^{-1} + (2 \times 3^{-4}) \dots = 1.357\dots = 1.36$  to two decimal places.

To check these, we have  $p = 201_3$  as a decimal number is  $(2 \times 3^2) + 3^0 = 19$  and  $q = 112_3$  is  $3^2 + 3^1 + (2 \times 3^0) = 14$ . Thus in the decimal system, (a)  $p + q = 19 + 14 = 33$ , (b)  $p - q = 19 - 14 = 5$ , (c)  $p \times q = 19 \times 14 = 266$ , and (d)  $p/q = 19/14 = 1.36$  to two decimal places, as required.

## 1.2 Real variables

The work in Section 1.1 can be generalised by representing real numbers as symbols,  $x$ ,  $y$ , etc. Thus we are entering the field of *algebra*. This section starts by generalising the methods of Section 1.1 for real numbers to algebraic quantities and also discusses the general idea of algebraic expressions and the important result known as the binomial theorem.

### 1.2.1 Rules of elementary algebra

Algebra enables us to consider general expressions like, for example,  $(x + y)^2$ , where  $x$  and  $y$  can be any real number. When manipulating real numbers as symbols, the fundamental rules of algebra apply. These are analogous to the basic rules of arithmetic given in Section 1.1 and can be summarised as follows.<sup>2</sup>

(i) *Commutativity*: Addition and multiplication are commutative operations, i.e.

$$x + y = y + x \quad \text{commutative law of addition} \quad (1.14a)$$

and

$$xy = yx \quad \text{commutative law of multiplication.} \quad (1.14b)$$

In contrast, subtraction and division are only commutative operations under special circumstances. Thus,

$$x - y \neq y - x \quad \text{unless } x = y$$

and

$$x \div y \neq y \div x \quad \text{unless } x = y \text{ and neither equals zero.}$$

(ii) *Associativity*: Addition and multiplication are associative operations, i.e.,

$$x + (y + z) = (x + y) + z \quad \text{associative law of addition} \quad (1.15a)$$

<sup>2</sup>Here and in what follows the explicit multiplication signs between terms are usually omitted if there is no loss of clarity, so that  $xy$  is equivalent to  $x \times y$  and so on.